

# Lattice Polytopes of Degree 2

Jaron Treutlein

**ABSTRACT.** A theorem of Scott gives an upper bound for the normalized volume of lattice polygons with exactly  $i > 0$  interior lattice points. We will show that the same bound is true for the normalized volume of lattice polytopes of degree 2 even in higher dimensions. In particular, there is only a finite number of quadratic polynomials with fixed leading coefficient being the  $h^*$ -polynomial of a lattice polytope.

## 1. Introduction

An  $n$ -dimensional lattice polytope  $P \subset \mathbb{R}^n$  is the convex hull of a finite number of elements of  $\mathbb{Z}^n$ . In the following, we denote by  $\text{Vol}(P) = n! \text{vol}(P)$  the normalized volume of  $P$  and may call it the volume of  $P$ . By  $\Pi^{(1)} := \Pi(P) \subset \mathbb{R}^{n+1}$ , we denote the convex hull of  $(P, 0) \subset \mathbb{R}^{n+1}$  and  $(0, \dots, 0, 1) \in \mathbb{R}^{n+1}$ , which we will call the standard pyramid over  $P$ . Recursively we define  $\Pi^{(k)}(P) = \Pi(\Pi^{(k-1)}(P))$  for all  $k > 0$ .  $\Delta_n$  will denote the  $n$ -dimensional basic lattice simplex throughout, i.e.  $\text{Vol}(\Delta_n) = 1$ . If two lattice polytopes  $P$  and  $Q$  of the same dimension are equivalent via some affine unimodular transformation, we will write  $P \cong Q$ . The  $k$ -fold of a polytope  $P$  will be the convex hull of the  $k$ -fold vertices of  $P$  for every  $k \geq 0$ .

Pick's formula gives a relation between the normalized volume, the number of interior lattice points and the number of lattice points of a lattice polygon, i.e. of a two-dimensional lattice polytope:  $\text{Vol}(P) = |P \cap \mathbb{Z}^2| + |P^\circ \cap \mathbb{Z}^2| - 2$ . Here  $P^\circ$  means the interior of the polytope  $P$ .

In 1976 Paul Scott [9] proved that the volume of a lattice polygon with exactly  $i \geq 1$  interior lattice points is constrained by  $i$ :

**THEOREM 1.1** (Scott). *Let  $P \subset \mathbb{R}^2$  be a lattice polygon such that  $|P^\circ \cap \mathbb{Z}^2| = i \geq 1$ . If  $P \cong 3\Delta_2$ , then  $\text{Vol}(P) = 9$  and  $i = 1$ . Otherwise the normalized volume is bounded by  $\text{Vol}(P) \leq 4(i + 1)$ . According to Pick's formula, this implies  $|P \cap \mathbb{Z}^2| \leq 3i + 6$  and  $|P \cap \mathbb{Z}^2| \leq \frac{3}{4}\text{Vol}(P) + 3$ .*

Besides Scott's proof, there are two proofs by Christian Haase and Joseph Schicho [5]. Another proof is given in [13].

Our aim is to generalize Scott's theorem. Therefore we need to introduce another invariant, the degree of a lattice polytope:

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It is known from [4], [10] and [11] that  $h_P^*(t) := (1-t)^{n+1} \sum_{k \geq 0} |kP \cap \mathbb{Z}^n| t^k \in \mathbb{Z}[t]$  is a polynomial of degree  $d \leq n$ . This number is described as the degree of  $P$  and is the largest number  $k \in \mathbb{N}$  such that there is an interior lattice point in  $(n+1-k)P$  (cf. [2]). The leading coefficient of  $h_P^*$  is the number of interior lattice points in  $(n+1-d)P$  and the constant coefficient is  $h_P^*(0) = 1$ . Moreover the sum of all coefficients is the normalized volume of  $P$  and all coefficients are non-negative integers by the non-negativity theorem of Richard P. Stanley [10].

It is easy to show that the  $h^*$ -polynomial of  $P$  and  $\Pi(P)$  are equal. So  $P$  and  $\Pi(P)$  have the same degree and the same normalized volume, which is the sum of all coefficients of the  $h^*$ -polynomial. Moreover

$$\left| \left( (n+2-d)\Pi(P) \right)^\circ \cap \mathbb{Z}^{n+1} \right| = \left| \left( (n+1-d)P \right)^\circ \cap \mathbb{Z}^n \right|.$$

Scott's theorem shows that the normalized volume of a two-dimensional lattice polytope of degree 2 with exactly  $i > 0$  interior lattice points is bounded by  $4(i+1)$ , except for one single polytope:  $3\Delta_2$ . We generalize this result to the case of  $n$ -dimensional lattice polytopes of degree 2.

**THEOREM 1.2.** *Let  $P \subset \mathbb{R}^n$  be a  $n$ -dimensional lattice polytope of degree 2. If  $P \cong \Pi^{(n-2)}(3\Delta_2)$ , then  $\text{Vol}(P) = 9$ ,  $|P \cap \mathbb{Z}^n| = 8 + n$  and  $\left| \left( (n-1)P \right)^\circ \cap \mathbb{Z}^n \right| = 1$ . Otherwise the following equivalent statements hold:*

- (1)  $\text{Vol}(P) \leq 4(i+1)$
- (2)  $b \leq 3i + n + 4$
- (3)  $b \leq \frac{3}{4}\text{Vol}(P) + n + 1$ ,

where  $b := |P \cap \mathbb{Z}^n|$  and  $i := \left| \left( (n-1)P \right)^\circ \cap \mathbb{Z}^n \right| \geq 1$ .

The following theorem of Victor Batyrev [1] motivates our estimation of the normalized volume of a lattice polytope of degree  $d$ :

**THEOREM 1.3 (Batyrev).** *Let  $P \subset \mathbb{R}^n$  be an  $n$ -dimensional lattice polytope of degree  $d$ . If*

$$n \geq 4d \binom{2d + \text{Vol}(P) - 1}{2d},$$

*then  $P$  is a standard pyramid over an  $(n-1)$ -dimensional lattice polytope.*

There is a recent result by Benjamin Nill [7] which even strenghtens this bound:

**THEOREM 1.4 (Nill).** *Let  $P \subset \mathbb{R}^n$  be a  $n$ -dimensional lattice polytope of degree  $d$ . If*

$$n \geq (\text{Vol}(P) - 1)(2d + 1),$$

*then  $P$  is a standard pyramid over an  $(n-1)$ -dimensional lattice polytope.*

Jeffrey C. Lagarias and Günter M. Ziegler showed in [6] that up to unimodular transformation there is only a finite number of  $n$ -dimensional lattice polytopes having a fixed volume. From Theorem 1.3 or Theorem 1.4 follows

**COROLLARY 1.5 (Batyrev).** *For a family  $\mathcal{F}$  of lattice polytopes of degree  $d$ , the following is equivalent:*

- (1)  $\mathcal{F}$  is finite modulo standard pyramids and affine unimodular transformation,

(2) There is a constant  $C_d > 0$  such that  $\text{Vol}(P) \leq C_d$  for all  $P \in \mathcal{F}$ .

CONJECTURE 1.6 (Batyrev). Let  $P$  be a lattice polytope of degree  $d$  with exactly  $i \geq 1$  interior lattice points in its  $(\dim(P) + 1 - d)$ -fold. Its normalized volume  $\text{Vol}(P)$  can then be bounded by a constant  $C_{d,i}$ , only depending on  $d$  and  $i$ . The finiteness of lattice polytopes of degree  $d$  with this property up to standard pyramids and affine unimodular transformation follows from Theorem 1.3.

Theorem 1.2 proves Conjecture 1.6 in the case  $d = 2$ .

COROLLARY 1.7. Up to affine unimodular transformations and standard pyramids there is only a finite number of lattice polytopes of degree 2 having exactly  $i \geq 1$  interior lattice points in their adequate multiple.

This follows from Theorem 1.2 and Theorem 1.3.

COROLLARY 1.8. There is only a finite number of quadratic polynomials  $h \in \mathbb{Z}[t]$  with leading coefficient  $i \in \mathbb{N}$ , such that  $h$  is the  $h^*$ -polynomial of a lattice polytope.

This follows from Theorem 1.2 and the fact that all coefficients of  $h_P^*$  are positive integers summing up to  $\text{Vol}(P)$ .

In the remaining part of the paper we prove Theorem 1.2.

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## 2. Preparations

The formula of Pick can be easily generalized for higher dimensional polytopes of degree 2 using their  $h^*$ -polynomial. This shows that statements (1) – (3) in Theorem 1.2 are equivalent.

LEMMA 2.1. *An  $n$ -dimensional lattice polytope of degree 2 has normalized volume  $\text{Vol}(P) = b + i - n$ , where  $b := |P \cap \mathbb{Z}^n|$  and  $i := \left| \left( (n-1)P \right)^\circ \cap \mathbb{Z}^n \right|$ .*

PROOF. Proof. The normalized volume of  $P$  can be computed by adding the coefficients of the  $h^*$ -polynomial of  $P$ . Consequently  $\text{Vol}(P) = 1 + (b - n - 1) + i$ .  $\square$

Let  $s \subset P$  be a face of  $P$ . By  $\text{st}(s) = \bigcup F$ , we denote the star of  $s$  in  $P$ , where the union is over all faces  $F \subset P$  of  $P$  containing  $s$ .

LEMMA 2.2. *Let  $P$  be an  $n$ -dimensional lattice polytope of degree 2 and  $s \subset P$  a face of  $P$  having exactly  $j > 0$  interior lattice points in its  $(n-2)$ -fold:*

$$\left( (n-2)s \right)^\circ \cap \mathbb{Z}^n = \{x_1, \dots, x_j\}.$$

Moreover, we suppose

$$z := \left| P \setminus \text{st}(s) \cap \mathbb{Z}^n \right| \geq 1.$$

Then  $0 < j + z - 1 \leq \left| \left( (n-1)P \right)^\circ \cap \mathbb{Z}^n \right|$ .

REMARK 2.3. Let us first consider an easy case.

If  $z = 1$ , i.e.  $P \setminus \text{st}(s) \cap \mathbb{Z}^n = \{p\}$ , then

$$p + x = (n-1) \left( \frac{n-2}{n-1} \frac{x}{n-2} + \frac{p}{n-1} \right) \in \left( (n-1)P \right)^\circ \cap \mathbb{Z}^n \quad \forall x \in \left( (n-2)s \right)^\circ \cap \mathbb{Z}^n$$

yield  $j > 0$  distinct lattice points in  $(n-1)P$ . So  $0 < j \leq \left| \left( (n-1)P \right)^\circ \cap \mathbb{Z}^n \right|$  as claimed.

PROOF. If  $l = 1$ , the claim is certainly correct. Hence let  $l \geq 2$ .

There is a lattice point  $z_l \in y_l^\perp \cap \left( D \setminus \left\{ \frac{x_1}{n-2} \right\} \right) \cap \mathbb{Z}^{n+1}$ . Define  $\pi_l := \text{conv}(s_l, z_l)$ . Obviously  $\pi_l \cap s = s_l$ . By induction, there are further pyramids  $\pi_1, \dots, \pi_{l-1}$  satisfying  $\pi_k \cap s = s_k$  and  $\pi_k \cap \pi_{k'} \subset \{z_1, \dots, z_{k'}\} \subset \partial D \cap \mathbb{Z}^{n+1} \quad \forall k < k' < l$ .

Assume  $\pi_l \cap \pi_k \not\subset \{z_1, \dots, z_l\}$ , i.e. there exists a point  $q \in \pi_l \cap \pi_k$ ,  $q \notin \{z_1, \dots, z_l\}$  and  $k < l$ . Therefore  $y_k|_{\pi_k} \geq 0$ , because  $y_k|_{s_k} \geq 0$  and  $y_k(z_k) = 0$ . In particular,  $y_k(q) \geq 0$ , and  $y_l(q) \geq 0$  as well. As  $q \in \pi_l = \text{conv}(s_l, z_l)$ , there is a point  $p \in s_l$  and a number  $\lambda \in [0, 1]$  such that  $q = \lambda p + (1 - \lambda)z_l$ . Therefore  $0 \leq y_k(q) = \lambda y_k(p) + (1 - \lambda)y_k(z_l)$  with  $y_k(z_l) \leq 0$  as  $z_l \in D$  and  $y_k|_D \leq 0$ .

If  $y_k(p) \geq 0$ , then  $p \in s \cap \{x \in \mathbb{R}^{n+1} : y_k(x) \geq 0\} = s'_k \subseteq \bigcup_{r \leq k} s_r$ . But this is a contradiction to  $p \in s_l$  with  $l > k$ . So

$$0 \leq y_k(q) = \lambda y_k(p) + (1 - \lambda)y_k(z_l) \leq 0$$

with equality only in the case of  $\lambda = 0$  and  $y_k(z_l) = 0$ . Therefore the intersection of  $\pi_k$  and  $\pi_l$  is  $q = z_l$  or empty. This is a contradiction to  $\pi_l \cap \pi_k \not\subset \{z_1, \dots, z_l\}$  and so the claim is proven.  $\square$

The pyramids  $\pi_1, \dots, \pi_K$  intersect with  $D$  only in faces of  $D$ .

To any  $k \in \{1, \dots, K\}$  denote by  $a_k := \left| \left( (n-2)s_k \right)^\circ \cap \mathbb{Z}^{n+1} \right|$  the number of interior lattice points of  $(n-2)s_k$ . By Remark 2.3, there are  $a_k \geq 0$  interior lattice points of  $(n-1)s$  in  $(n-1)\pi_k$ . By adding up the number of interior lattice points in  $(n-1)\pi_1, \dots, (n-1)\pi_K$ , we derive from the claim

$$\left| \bigcup_{k=1}^K \left( (n-1)\pi_k \right)^\circ \cap \mathbb{Z}^{n+1} \right| \geq \sum_{k=1}^K a_k = j - 1.$$

Furthermore to every  $p \in D \setminus \left\{ \frac{x_1}{n-2} \right\}$  we get a lattice point of  $\left( (n-1) \left( D \setminus \left\{ \frac{x_1}{n-2} \right\} \right) \right)^\circ \subset \left( (n-1)P \right)^\circ$  in the following way:

$$p + x_1 = (n-1) \left( \frac{n-2}{n-1} \frac{x_1}{n-2} + \frac{p}{n-1} \right) \in \left( (n-1)D \right)^\circ \cap \mathbb{Z}^{n+1}.$$

Finally we get  $\left| \left( (n-1)P \right)^\circ \cap \mathbb{Z}^{n+1} \right| \geq j - 1 + z$ .  $\square$

### 3. The Proof of the Main Theorem

If  $n = 2$ , then Theorem 1.2 is equal to Scott's Theorem 1.1. So let  $n > 2$ .

The monotonicity theorem of Stanley [12] says that the degree of every face of a polytope is not greater than the degree of the polytope itself. In particular this is true for every facet. So we will distinguish the two cases that there is a facet of  $P$  having degree 2 or there is not.

For the second case we need a result of Victor Batyrev and Benjamin Nill. They

proved in [2] that every  $n$ -dimensional lattice polytope of degree less than 2 either is equivalent to a pyramid over the exceptional lattice simplex  $2\Delta_2$  or it is a Lawrence polytope, i.e. a lattice polytope projecting along an edge onto an  $(n-1)$ -dimensional basic simplex.

**Case 1:** There is a facet  $F \subset P$  of  $P$  having degree two, i.e.

$$\left| \left( (n-2)F \right)^\circ \cap \mathbb{Z}^n \right| = j \geq 1.$$

Define  $z := |P \setminus F \cap \mathbb{Z}^n|$ . From Lemma 2.2 we get  $z + j - 1 \leq i$ . Thus, by induction, we get, if  $F \not\cong \Pi^{(n-3)}(3\Delta_2)$ ,

$$\begin{aligned} |P \cap \mathbb{Z}^n| &= |F \cap \mathbb{Z}^n| + |(P \setminus F) \cap \mathbb{Z}^n| \leq 3j + n - 1 + 4 + z \\ &= 3(j + z - 1) - 2z + 2 + n + 4 \stackrel{z \geq 1}{\leq} 3i + n + 4, \end{aligned}$$

Otherwise  $F \cong \Pi^{(n-3)}(3\Delta_2)$  and again by induction and Lemma 2.2:  $|F \cap \mathbb{Z}^n| = (n-1) + 8$ ,  $z \leq i$  and so  $|P \cap \mathbb{Z}^n| = n - 1 + 8 + z \leq i + 7 + n$ . This term is smaller than  $3i + n + 4$  if  $i \geq 2$ . If  $i = 1$  however, we get

$$n + 8 \leq |P \cap \mathbb{Z}^n| = n + 7 + z \leq i + 7 + n = 8 + n,$$

so  $|P \cap \mathbb{Z}^n| = 8 + n$  and  $\text{Vol}(P) = 9$  by Lemma 2.1. In this case  $P \cong \Pi^{(n-2)}(3\Delta_2)$  because  $\text{Vol}(F) = 9$  and  $F \cong \Pi^{(n-3)}(3\Delta_2)$ .

**Case 2:** Every facet  $F$  of  $P$  has degree  $\deg(F) \leq 1$ .

Let  $y$  be an edge of  $P$  having the maximal number of lattice points; its length will be denoted by  $h_1$ , i.e.  $h_1 = |y \cap \mathbb{Z}^n| - 1$ . Among all 2-codimensional faces of  $P$  containing  $y$ ,  $s$  should be the face having the maximal number of lattice points. We will denote by  $F_1$  and  $F_2$  the two facets of  $P$  containing  $s$ .

Again the monotonicity theorem of Stanley [12] implies  $\deg(s) \leq \deg(F_1) = 1$ . Similarly to case 1, we will denote by  $z := |P \setminus \{F_1 \cup F_2\} \cap \mathbb{Z}^n|$  the number of lattice points of  $P$  not in  $F_1$  and  $F_2$ .

By the result of Victor Batyrev and Benjamin Nill [2] we find that the facets  $F_1$  and  $F_2$  are either  $(n-1)$ -dimensional Lawrence polytopes or pyramids over  $2\Delta_2$ .

(A)  $F_1$  and  $F_2$  are Lawrence polytope with heights  $h_1^{(k)}, h_2^{(k)}, \dots, h_{n-1}^{(k)} \forall k \in \{1, 2\}$ , where we assume that  $h_l^{(1)} = h_l^{(2)} = h_l \forall l \in \{1, \dots, n-2\}$ ,

$$s = \text{conv}(0, h_1 e_1, e_l, e_l + h_l e_1 : 2 \leq l \leq n-2),$$

where  $\{e_1, \dots, e_{n-2}, e_{n-1}^{(k)}\}$  should denote a lattice basis of  $\text{lin}(F_k) \cap \mathbb{Z}^n$  such that  $F_k = \text{conv}(s, e_{n-1}^{(k)}, e_{n-1}^{(k)} + h_{n-1}^{(k)} e_1)$  for  $k \in \{1, 2\}$ . Since the degree of the Lawrence prism  $s$  is at most one, we obtain

$$\left| \left( (n-2)s \right)^\circ \cap \mathbb{Z}^n \right| = \text{Vol}(s) - 1 = \left( \sum_{l=1}^{n-2} h_l \right) - 1.$$

We may assume  $z = |(P \setminus \{F_1 \cup F_2\}) \cap \mathbb{Z}^n| \neq 0$  because otherwise  $P$  would be a prism over the face  $P \cap \{X_1 = 0\}$ , which is an  $(n-1)$ -dimensional lattice simplex of degree at most 1, whose only lattice points are vertices. By [2] this is a basic simplex and

hence  $P$  is a Lawrence polytope. Consequently  $\deg(P) < 2$ , a contradiction. We have to distinguish the following two cases:

$$(i) \left| \left( (n-2)s \right)^\circ \cap \mathbb{Z}^n \right| \geq 1.$$

Because of Lemma 2.2, we get the estimation

$$z + \left( \left( \sum_{l=1}^{n-2} h_l \right) - 1 \right) - 1 \leq i.$$

So we can bound the number of lattice points of  $P$ :

$$\begin{aligned} |P \cap \mathbb{Z}^n| &= |(F_1 \cup F_2) \cap \mathbb{Z}^n| + z = |s \cap \mathbb{Z}^n| + h_{n-1}^{(1)} + 1 + h_{n-1}^{(2)} + 1 + z \\ &= \sum_{l=1}^{n-2} h_l + (n-2) + h_{n-1}^{(1)} + h_{n-1}^{(2)} + 2 + z \leq i + n + 2h_1 + 2 \\ &\stackrel{h_1 \leq i+1}{\leq} i + n + 2(i+1) + 2 = 3i + n + 4. \end{aligned}$$

$$(ii) \left| \left( (n-2)s \right)^\circ \cap \mathbb{Z}^n \right| = 0.$$

In this case,  $s$  has degree zero, so it is a basic simplex. Our assumption on  $s$  implies that every lattice point of  $P$  is a vertex. If  $n = 3$ , then Howe's theorem [8] yields that  $P$  has at most 8 vertices, therefore  $|P \cap \mathbb{Z}^n| \leq 8 < n + 4 + 3i$ . So let  $n \geq 4$ .

In that case, since every 2-codimensional face is a simplex and every facet is a Lawrence prism, we see that  $P$  is simplicial, i.e. every facet is a simplex. We may suppose that  $P$  is not a simplex. Let  $S$  be a subset of the vertices of  $P$  such that the convex hull of  $S$  is not a face of  $P$ . Then the sum over the vertices of  $S$  is a lattice point in the interior of  $|S| \cdot P$ . Since the degree of  $P$  is two, this implies  $|S| \geq n-1$ . In other words, every subset of the vertices of  $P$  that has cardinality at most  $n-2$  forms the vertex set of a face of  $P$ , i.e.  $P$  is  $(n-2)$ -neighbourly. As is known from [3], a polytope of dimension  $n$  that is not a simplex is at most  $\lfloor \frac{n}{2} \rfloor$ -neighbourly. Therefore  $n-2 \leq \frac{n}{2}$ . This shows  $n = 4$ .

Let  $f_j \geq 0$  be the number of  $j$ -dimensional faces of  $P$ . Since  $P$  is a 2-neighbourly simplicial 4-dimensional polytope we get  $f_1 = \binom{f_0}{2}$  and  $f_2 = 2f_3$ . Since the Euler characteristic of the boundary of  $P$  vanishes, i.e.  $f_0 - f_1 + f_2 - f_3 = 0$ , we deduce  $f_3 = \frac{f_0(f_0-3)}{2}$ . Let  $\mathcal{D}$  denote the set of subsets  $\Delta$  of the vertices of  $P$  such that  $\Delta$  has cardinality three but  $\Delta$  is not the vertex set of a face of  $P$ . Therefore,  $|\mathcal{D}| = \binom{f_0}{3} - f_2 = f_0 \left( \frac{(f_0-1)(f_0-2)}{6} - (f_0-3) \right)$ . Since  $|\{(e, \Delta) : e \text{ is an edge of } P, \Delta \in \mathcal{D}, e \subset \Delta\}| = 3|\mathcal{D}|$ , double counting yields that there exists an edge  $e$  of  $P$  that is contained in at least  $\frac{3|\mathcal{D}|}{f_1}$  many elements  $\Delta \in \mathcal{D}$ . Therefore, any such  $\Delta$  contains one vertex that is not in the star of  $e$ , and hence Lemma 2.2 yields

$$i \geq \frac{3|\mathcal{D}|}{f_1} = f_0 - 2 - 6 \frac{f_0 - 3}{f_0 - 1} \geq f_0 - 8.$$

Thus,  $|P \cap \mathbb{Z}^n| = f_0 \leq 8 + i < n + 4 + 3i$ .

(A')  $F_1$ ,  $F_2$  and  $s$  have no common projection direction.

Without loss of generality let  $F_1$  and  $s$  have two different projection directions. If  $s$  contains an edge of length at least 2, then this has to be a common projection

direction with  $F_1$ , because  $s$  and  $F_1$  are Lawrence prisms. But this is a contradiction. Hence, all lattice points in  $s$  are vertices. In particular,  $y$  has length one, so also all lattice points of  $P$  are vertices.

Since any of the two different projection directions of the Lawrence prism  $s$  maps a four-gon face onto the edge of an unimodular base simplex and two edges of the four-gon give the projection direction, we see that there is at most one four-gon face in  $s$ . Therefore,  $s$  contains at most  $(n-2) + 2 = n$  lattice points.

Since  $F_k$  contains at most two vertices not in  $s$  for  $k \in \{1, 2\}$ , we get  $|(F_1 \cup F_2) \cap \mathbb{Z}^n| \leq n + 4 + n + 4 + 3i$ . Therefore we may assume  $z := |P \setminus (F_1 \cup F_2) \cap \mathbb{Z}^n| \neq 0$ .

If  $\left| \left( (n-2)s \right)^\circ \cap \mathbb{Z}^n \right| = 0$ , then we will proceed exactly like in case (ii) from (A).

So let  $j := \left| \left( (n-2)s \right)^\circ \cap \mathbb{Z}^n \right| \geq 1$ .

Because of Lemma 2.2, we get the estimation  $z + j - 1 \leq i$ , in particular  $z \leq i$ . Hence we can bound the number of lattice points of  $P$ :

$$|P \cap \mathbb{Z}^n| = |(F_1 \cup F_2) \cap \mathbb{Z}^n| + z \leq n + 4 + i < 3i + n + 4.$$

**(B)**  $F_1$  is a Lawrence polytope with the heights  $h_1 \geq h_2 \geq \dots \geq h_{n-1}$ ,  $F_2 \cong \Pi^{(n-3)}(2\Delta_2)$ .

Here

$$s \cong \text{conv}(0, h_1 e_1, e_l, 2 \leq l \leq n-2)$$

and  $h_1 = 2$ ,  $h_2 = \dots = h_{n-2} = 0$ , because  $s$  is contained in the simplex  $F_2$ . If  $z = |P \setminus \{F_1 \cup F_2\} \cap \mathbb{Z}^n| = 0$ , then

$$\begin{aligned} |P \cap \mathbb{Z}^n| &= |F_2 \cap \mathbb{Z}^n| + |F_1 \setminus F_2 \cap \mathbb{Z}^n| = 6 + (n-3) + h_{n-1} + 1 \\ &\stackrel{h_{n-1} \leq h_1=2}{\leq} 4 + n + 2 < 3i + n + 4. \end{aligned}$$

Otherwise if  $z \geq 1$ , we obtain just like in (A)  $0 < z + (h_1 - 1) - 1 \leq i$ . Therefore

$$\begin{aligned} |P \cap \mathbb{Z}^n| &= |(F_1 \cup F_2) \cap \mathbb{Z}^n| + z = |s \cap \mathbb{Z}^n| + (h_{n-1} + 1) + 3 + z \\ &= h_1 + (n-2) + (h_{n-1} + 1) + 3 + z \leq i + 4 + h_{n-1} + n \\ &\stackrel{h_{n-1} \leq h_1=2}{\leq} 3i + n + 4. \end{aligned}$$

**(C)**  $F_1 \cong F_2 \cong \Pi^{(n-3)}(2\Delta_2)$ .

Here either  $s$  is a pyramid over  $2\Delta_1$  or  $s \cong \Pi^{(n-4)}(2\Delta_2)$ . Again  $h_1 = 2$ .

If  $z = |P \setminus \{F_1 \cup F_2\} \cap \mathbb{Z}^n| = 0$ , then

$$|P \cap \mathbb{Z}^n| = |F_2 \cap \mathbb{Z}^n| + |F_1 \setminus F_2 \cap \mathbb{Z}^n| \leq 6 + (n-3) + 3 < 3i + n + 4.$$

Otherwise if  $z \geq 1$ , we obtain  $z \leq i$  because of  $\left| \left( (n-2)s \right)^\circ \cap \mathbb{Z}^n \right| \geq 1$  and Lemma 2.2. So as a result

$$\begin{aligned} |P \cap \mathbb{Z}^n| &= |F_1 \cap \mathbb{Z}^n| + |F_2 \setminus F_1 \cap \mathbb{Z}^n| + z \leq (6 + n - 3) + 3 + z = n + z + 6 \\ &\leq n + i + 6 \leq n + 3i + 4. \end{aligned}$$

This completes the proof.  $\square$

REMARK 3.1. In [11], Stanley shows that the coefficients of  $h_P^*$  also appear in the polynomial  $(1-t)^{n+1} \sum_{k \geq 0} |(kP)^\circ \cap \mathbb{Z}^n| t^k \in \mathbb{Z}[t]$ . So we can also compute the coefficients of  $h_P^*$  in a different way than in Lemma 2.1. Then it is easy to show that the bounds of Theorem 1.2 are also equivalent to the following estimations:

$$\begin{aligned} |(nP)^\circ \cap \mathbb{Z}^n| &\leq (n+4)i + 3, \\ |2P \cap \mathbb{Z}^n| &\leq (4+3n)(i+1) + \frac{n(n+3)}{2}. \end{aligned}$$

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DEPARTMENT OF MATHEMATICS AND PHYSICS, UNIVERSITY OF TÜBINGEN, AUF DER MORGENSTELLE 10, D-72076 TÜBINGEN, GERMANY

*E-mail address:* jaron@mail.mathematik.uni-tuebingen.de